JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **169**, 359–373 (1992)

A Class of Nonconvex Functions and Pre-variational Inequalities

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Submitted by C Foias

Received February 1, 1991

A class of nonconvex functions is introduced, called semi-preinvex function, which includes the classes of preinvex functions and arc-connected convex functions. The Fritz-John conditions of the mathematical programming problem are derived for these kinds of functions. The pre-variational inequality is given as a necessary condition and also a sufficient condition for a mathematical programming for invex functions. The Type I function related to unconstrained problems is given as an equivalent form of the pre-variational inequality. Existence theorems for the solution of the pre-variational inequality are also proved.

1. INTRODUCTION

A significant generalization of convex functions is the introduction of arc-connected convex functions (see, for example, [9]). Recently another meaningful generalization of convex functions, invex functions, was given by M. A. Hanson [4] and B. D. Craven [2]. Much significant work has been done for this kind of nonconvex function (see [1–8] and the references therein). Various kinds of necessary and sufficient conditions in which the invex and preinvex functions (see (1) and (2)) were closely involved have been derived. Both the preinvex functions and the arc-connected convex functions (see (3)) are generalizations of convex functions. It appears

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0022-247X/92 $5.00

from the definitions that the preinvex functions and arcwise connected convex functions have some similar characteristics. However, it is easy to verify that these are indeed two different classes of functions but share some similar properties. So it is the purpose of this paper to introduce a wider class of nonconvex functions—*semi-preinvex*— which includes the preinvex function and arcwise connected convex functions and preserves some nice properties that convex functions have.

As some applications of semi-preinvex functions, (i) the Fritz-John condition for the inequality constrained optimization problem is derived by using an arcwise directionally differentiable assumption and the alternative theorem for convex-like functions (see [7]) since the semi-preinvex function is also a convex-like function; (ii) the pre-variational inequality problem, which includes the Fritz-John condition as a special case, is introduced and the concept of Type I function for the unconstrained mathematical programming problem is given as an equivalent form of the pre-variational inequality problem (see Definition 2). This follows from Hanson and Mond [5] and Rueda and Hanson [6], who proved the relations between the Kuhn–Tucker point of "Type I and Type II functions" and a minimum of a constrained mathematical programming problem (see [5, 6]). We prove that the stable point of a Type I function is a minimum of an unconstrained mathematical programming problem.

Finally, existence theorems of the pre-variational inequality are proved under some conditions (see Section 4). The Knaster–Kuratowski–Mazurkiewicz theorem (see [1]) is a powerful tool in many fields, such as fixed point theory and variational inequality problems. Mosco [12] used the Knaster–Kuratowski–Mazurkiewcz theorem to prove the existence of the usual variational inequality. In Section 4, we employ the Knaster–Kuratowski–Mazurkiewcz theorem to discuss the pre-variational inequality problem and we give some restrictions on the function "$\tau(y, x)$" (pre-coercive conditions, normality, regularity) which, compared to the conditions given in [10, 12, 13], are reasonable.

2. Semi-preinvex Functions and Mathematical Programming

Let R^n denote the n-dimension Euclidean space. In [4], M. A. Hanson considered the real differentiable function $f(x)$ on R^n with the gradient $\nabla f(x)$ which satisfies: for any x, $y \in R^n$, there exists a vector $\tau(y, x) \in R^n$ such that

$$f(y) \geqslant f(x) + \nabla f(x)\, \tau(y, x). \tag{1}$$

B. D. Craven [2] called it invex function. Later, a characteristic of the invex function was given by A. Ben-Israel and B. Mond [1], that is, a real differentiable function f: $K \subset R^n \to R$ is invex if, for any x, $y \in K$, there exists a vector $\tau(y, x) \in R^n$, s.t. $\forall \alpha \in [0, 1]$, $x + \alpha\tau(y, x) \in K$ and

$$f(x + \alpha\tau(y, x)) \leqslant \alpha f(y) + (1 - \alpha) f(x). \tag{2}$$

(Note that in [1], $K = R^n$.) Such a function is called a preinvex function with respect to $\tau(y, x)$ and the property "for each x, $y \in K$, $\alpha \in [0, 1]$, $x + \alpha\tau(y, x) \in K$" is called "$\tau$-connectedness" in T. Weir and V. Jeyakumar [8].

A subset C is said to be arc connected [9] if, for every pair of points x, $y \in C$, there exists a continuous arc $H(y, x, \alpha)$ defined on $[0, 1]$ with a value in C such that

$$H(y, x, 0) = x, \qquad H(y, x, 1) = y.$$

The function f: $C \to R$ is said to be arc-connected convex on an arcwise connected set C of R^n [9] if for any x, $y \in C$, $\alpha \in [0, 1]$

$$f(H(y, x, \alpha)) \leqslant \alpha f(y) + (1 - \alpha) f(x) \tag{3}$$

holds.

EXAMPLE 1. An arcwise connected set is not the same as a "τ-connected" set. Let

$$A_1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leqslant 1\},$$
$$A_2 = \{(x_1, x_2) \mid (x_1 - 3)^2 + x_2^2 \leqslant 1\}.$$

$A = A_1 \cup A_2$ is a "τ-connected" set if we choose

$$\begin{aligned} \tau(y, x) &= y - x \qquad \text{if both } x, y \in A_1 \text{ or } A_2; \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

But A is not an arcwise connected set. Let

$$B = \{(x_1, x_2) \mid x_2 \geqslant 0, (x_1 - 1)^2 + x_2^2 = 1\}.$$

B is an arcwise connected set but not a τ-connected set if $\tau(y, x) \neq 0$.

EXAMPLE 2 [8]. A real function defined by $f(x) = -|x|$ is not an arcwise connected convex function on $(-\infty, +\infty)$, but it is a preinvex function if we choose

$$\begin{aligned} \tau(y, x) &= y - x \qquad \text{if } x \leqslant 0, y \leqslant 0; \text{ or } x > 0, y > 0; \\ &= x - y \qquad \text{otherwise.} \end{aligned}$$

Next a class of nonconvex functions which includes the classes of preinvex functions and arcwise connected convex functions is introduced.

A set K in R^n is said to satisfy the "semi-connected" property, if for any x, $y \in K$ and $\alpha \in [0, 1]$, there exists a vector $\tau(y, x, \alpha) \in R^n$, such that $x + \alpha\tau(y, x, \alpha) \in K$.

DEFINITION 1. Let K be a set in R^n having the "semi-connected" property with $\tau(y, x, \alpha)$: $K \times K \times [0, 1] \to R^n$ and $f(x)$ be a real function on K. Then f is called semi-preinvex with respect to $\tau(y, x, \alpha)$ if for x, $y \in K$ and $\alpha \in [0, 1]$,

$$f(y + \alpha\tau(y, x, \alpha)) \leqslant \alpha f(y) + (1 - \alpha) f(x) \tag{3'}$$

holds and $\lim_{\alpha \downarrow 0} \alpha\tau(y, x, \alpha) = \mathbf{0}$.

Both A and B in Example 1 are "semi-connected" sets. In general, we have that the arcwise connected set and the τ-connected subset are semi-connected sets.

THEOREM 1. (i) *A preinvex function with respect to* $\tau(y, x)$ *is a semi-preinvex function with respect to* $\tau(y, x, \alpha) = \tau(y, x)$.

(ii) *An arcwise connected convex function is a semi-preinvex function.*

Proof. (i) We simply choose $\tau(y, x, \alpha) = \tau(y, x)$.

(ii) Let f: $K \to R$ be an arcwise connected convex function on an arcwise connected set $K \subset R^n$. We choose $\tau(y, x, \alpha)$ as follows. Fix x, $y \in K$; then by arcwise connectedness of K, there exists a continuous arc, $H(y, x, \alpha)$ on $[0, 1]$, s.t. $H(y, x, 0) = x$, and $H(y, x, 1) = y$, such that

$$f(H(y, x, \alpha)) \leqslant \alpha f(y) + (1 - \alpha) f(x), \qquad \alpha \in [0, 1].$$

For $\alpha \in (0, 1]$, let

$$\tau(y, x, \alpha) = 1/\alpha(H(y, x, \alpha) - x).$$

Therefore

$$f(x + \alpha\tau(y, x, \alpha)) \leqslant \alpha f(y) + (1 - \alpha) f(x), \qquad \alpha \in [0, 1]$$

and $\lim_{\alpha \downarrow 0} \alpha\tau(y, x, \alpha) = \mathbf{0}$. Hence f is a semi-preinvex function with respect to $\tau(y, x, \alpha)$. ∎

THEOREM 2. *Let K be a semi-connected subset of R^n and f: $K \to R$ be a semi-preinvex function with respect to $\tau(x, y, \alpha)$. Then any local minimum of f is a global minimum of f over K.*

Proof. Let x_0 be a local minimum of f on K. Assume that there exists $x \in K$ such that $f(x) < f(x_0)$; by the definition of semi-preinvexity, we have

$$f(x_0 + \alpha\tau(x, x_0, \alpha)) \leqslant \alpha f(x) + (1-\alpha) f(x_0), \qquad \forall \alpha \in [0, 1],$$

i.e,

$$f(x_0 + \alpha\tau(x, x_0, \alpha)) - f(x_0) \leqslant \alpha[f(x) - f(x_0)] < 0. \tag{4}$$

Let α be small enough and note that $x_0 + \alpha\tau(x, x_0, \alpha) \in K$ and $\lim_{\alpha \downarrow 0} \alpha\tau(x, x_0, \alpha) = \mathbf{0}$; then (4) contradicts the local minimum. ∎

Next, we derive the Fritz-John condition of the inequality constrained optimization problem to present a further application of semi-preinvex functions.

Consider the inequality constrained optimization problem:

(P_1) Minimize $f(x)$ subject to $x \in C$, $g_i(x) \leqslant 0$, $i = 1, ..., m$, where $C \subset R^n$, f, g_i $(i = 1, ..., m)$ are real functions on R^n.

Since the semi-preinvex function is also a convex-like function [7], that is, for any $x, y \in C$, $\alpha \in [0, 1]$, there exists $z \in C$ such that

$$f(z) \leqslant \alpha f(y) + (1-\alpha) f(x),$$

the following alternative theorem is an immediate direct result of Theorem 2 in [7].

LEMMA 1. *Let $h_i(x)$ $(i = 1, ..., k)$ be semi-preinvex functions. Then exactly one of the following two systems is solvable*:

(1) *there exists $x \in C$, $h_1(x) < 0, ..., h_k(x) < 0$;*

(2) *there exist $\lambda \in R^k_+ \backslash \{0\}$, $\lambda_i \geqslant 0$ $(i = 1, ..., k)$, $\sum_{(k)} \lambda_i h_i(C) \subset R_+$.*[1]

The function f: $C \to R$ is said to be arcwise directionally differentiable at $x_0 \in R^n$ if

$$f'(x_0; h) = \lim_{t \to 0} t^{-1}[f(x_0 + \omega(t)) - f(x_0)]$$

exists for each continuous arc ω: $[0, 1] \to R^n$ such that $\omega(0) = 0$, $\omega'(0^+) = h$.

[1] Throughout this paper, $\sum_{(k)} a_i$ always means that the sum of the element a_i is from a_1 to a_k.

THEOREM 3. *Consider the problem* (P_1). *Assume that* f *and* g_i $(i=1, \ldots, m)$ *are arcwise directionally differentiable and semi-preinvex functions with respect to the same* $\tau(y, x, \alpha)$ *satisfying* (H): $[d/d\alpha](\alpha\tau(y, x, \alpha))|_{\alpha=0}=\hat{\tau}(y, x)$ *for any* $x, y \in C$. *If* x_0 *is a minimum of* (P_1), *then there exist Lagrange multipliers* $\tau \geqslant 0$, $\lambda_i \geqslant 0$ $(i=1, \ldots, m)$, *not all zero, such that*

$$\tau f'(x_0; \hat{\tau}(x, x_0)) + \sum_{(m)} \lambda_i g_i'(x_0; \hat{\tau}(x, x_0)) \geqslant 0, \qquad \forall x \in C,$$

$$\lambda_i g_i(x_0) = 0 \ (i=1, \ldots, m). \tag{5}$$

Proof. Since x_0 is a minimum of (P_1), then the system

$$\text{there exists } x \in C, \qquad \text{such that} \qquad f(x)-f(x_0)<0,\ g_i(x)<0\ (i=1, \ldots, m),$$

has no solution. From Lemma 1, there exist $\tau \geqslant 0$, $\lambda_i \geqslant 0$ $(i=1, \ldots, m)$, not all zero, such that

$$\tau(f(x)-f(x_0)) + \sum_{(m)} \lambda_i g_i(x) \geqslant 0, \qquad \forall x \in C.$$

Let $x=x_0$; we get $\sum_{(m)} \lambda_i g_i(x_0) \geqslant 0$. Since $\lambda_i \geqslant 0$, $g_i(x_0) \leqslant 0$ $(i=1, \ldots, m)$, we get $\sum_{(m)} \lambda_i g_i(x_0)=0$. Since C is semi-connected, for any $x \in C$, $\alpha \in (0, 1]$, $x_0+\alpha\tau(x, x_0, \alpha) \in C$, we get

$$\tau(f(x_0+\alpha\tau(x, x_0, \alpha))-f(x_0)) + \sum_{(m)} \lambda_i (g_i(x_0+\alpha\tau(x, x_0, \alpha))-g_i(x_0)) \leqslant 0.$$

It follows from the arc directionally differentiability of f and g_i $(i=1, \ldots, m)$ and $(d/d\alpha)[\alpha\tau(y, x, \alpha)]|_{\alpha=0}=\hat{\tau}(y, x)$ that

$$\tau f'(x_0; \hat{\tau}(x, x_0)) + \sum_{(m)} \lambda_i g_i'(x_0; \hat{\tau}(x, x_0)) \geqslant 0, \qquad \forall x \in C,$$

$$\lambda_i g_i(x_0) = 0 \ (i=1, \ldots, m).$$

The proof is completed. ∎

Remark. If f and g_i $(i=1, \ldots, m)$ are preinvex functions with respect to $\tau(y, x)$, then the assumption (H) is trivially satisfied, since $(d/d\alpha)[\alpha\tau(y, x, \alpha)]|_{\alpha=0}=(d/d\alpha)[\alpha\tau(y, x)]|_{\alpha=0}=\tau(y, x)$. If f and g_i $(i=1, \ldots, m)$ are arc-connected convex functions, the assumption (H) is equivalent to

$$(d/d\alpha)[H(y, x, \alpha)]|_{\alpha=0}=\hat{\tau}(y, x).$$

3. Pre-variational Inequality

A well-known fact in mathematical programming is that the variational inequality problem has a close relation with the optimization problem. In this section, we introduce a class of variational inequalities—pre-variational inequalities which have a close relation with invex functions as we show below. The relations between pre-variational inequalities and other functions are also verified.

Let us recall the usual variational inequality problem: let K be a subset in R^n and T a mapping from R^n to R^n, and the varitional inequality is

$$\text{Find } x \in K, \qquad \text{such that} \qquad T(x)^t\,(y-x) \geqslant 0, \qquad \forall y \in K. \tag{6}$$

We refer to [10, 12, 14] for discussions of this variational inequality problem.

Following the model of the above variational inequality, we introduce the pre-variational inequality problem. Let $K \subset R^n$, T: $K \to R^n$, $\tau(y, x)$: $K \times K \to R^n$, f: $K \to R$, and the pre-variational inequality problem is

$$\text{Find } x \in K, \qquad \text{such that} \qquad T(x)^t\,\tau(y, x) \geqslant f(x) - f(y), \qquad \forall x \in K. \tag{7}$$

If $f = 0$, then (7) becomes

$$\text{Find } x \in K, \qquad \text{such that} \qquad T(x)^t\,\tau(y, x) \geqslant 0, \qquad \forall y \in K. \tag{8}$$

It is clear that (6) is a special case of (8) if $\tau(y, x) = y - x$. The next example shows the significance of the pre-variational inequality problem.

Example 3 (Fritz-John condition). Consider the problem (P_1) in Section 2. From Theorem 3, if $x_0 \in C$ is a minimum of (P_1), then there exist Lagrange multipliers $\tau \geqslant 0$, $\lambda_i \geqslant 0$ $(i = 1, ..., m)$, not all zero, such that

$$\tau f'(x_0; \hat{\tau}(x, x_0)) + \sum_{(m)} \lambda_i\, g_i'(x_0; \hat{\tau}(x, x_0)) \geqslant 0, \qquad \forall x \in C. \tag{5}$$

If f and g_i are differentiable at x_0, then (5) becomes

$$\left(\tau\, \nabla f(x_0) + \sum_{(m)} \lambda_i\, \nabla g_i(x_0)\right)^t \hat{\tau}(x, x_0) \geqslant 0, \qquad \forall x \in C.$$

This is a pre-variational inequality.

The next theorem presents a further application of the pre-variational inequality and shows the close relation between the pre-variational inequality and the invex functions. A special case of Theorem 4 can be found in Chipot [10].

Consider the unconstrained optimization problem

(P_2) Minimize $f(x)$, $\forall x \in K$, where K is a subset of n dimension Euclidean space R^n, f: $K \to R$.

THEOREM 4. *Let K be a subset in R^n having τ-connectedness, $x_0 \in K$, and let f be differentiable at x_0 and $T(x_0) = \nabla f(x_0)$. Then the following two statements hold*:

(i) *If x_0 is a minimum of* (P_2), *then x_0 is a solution of* (8);

(ii) *If f is an invex function with respect to $\tau(y, x)$ (see* (1)*), x_0 is a solution of* (8), *then x_0 is a minimum of* (P_2).

Proof. (i) Suppose that $x_0 \in K$ is a minimum of the problem (P_2). Then for each $y \in K$, $\alpha \in (0, 1]$, $x_0 + \alpha\tau(y, x_0) \in K$,

$$f(x_0 + \alpha\tau(y, x_0)) - f(x_0) \geqslant 0, \qquad \forall y \in K. \tag{9}$$

Since f is differentiable at x_0, dividing (9) by α, and then letting $\alpha \downarrow 0$, we get

$$\nabla f(x_0)' \, \tau(y, x_0) \geqslant 0, \qquad \forall y \in K.$$

Then (i) holds.

(ii) Suppose that x_0 is a solution of (8), i.e.,

$$T(x_0)' \, \tau(y, x_0) \geqslant 0, \qquad \forall y \in K.$$

From the definition of invex function, we have

$$\begin{aligned} f(y) &\geqslant f(x_0) + \nabla f(x_0)' \, \tau(y, x_0), \\ &\geqslant f(x_0), \qquad \forall y \in K. \end{aligned}$$

Therefore x_0 is a minimum of (P_2). ∎

Following Hanson and Mond [5] and Rueda and Hanson [6], we define the Type I function for the unconstrained optimization problem (P_2).

DEFINITION 2. f: $K \to R$ is a Type I function with respect to $\eta(x)$ at $x_0 \in K$ if there exists an n-dimensional vector function $\eta(x)$ defined on K such that

$$f(x) - f(x_0) \geqslant \nabla f(x_0)' \, \eta(x), \qquad \forall x \in K. \tag{10}$$

THEOREM 5 (Rueda and Hanson [6]). *If one of the following conditions holds, then $f(x)$ is a Type I function.*

(i) *There exists an n-dimensional vector function $\eta(x)$ such that*

$$f(y+\alpha\eta(x)) \leqslant \alpha f(x) + (1-\alpha) f(y), \qquad \alpha \in [0, 1], \forall x, y \in K.$$

(ii) *$f(x)$ is a convex function.*

In [6], Rueda and Hanson showed if the Kuhn–Tucker conditions are satisfied by the "Type I function," then the Kuhn–Tucker point is a minimum. In the rest of this section, we show that Type I functions are equivalent to pre-variational inequalities and that the stable point of a Type I function is a minimum. From the definition, it is easy to prove the following theorem.

THEOREM 6. *Let $x_0 \in K$, $T(x_0) = -\nabla f(x_0)$. Then the following two statements are equivalent:*

(i) *x_0 is a solution of pre-variational inequality* (7);

(ii) *$f(x)$ is a Type I function with respect to $\eta(x) = \tau(x, x_0)$ at x_0.*

Proof. (i) ⇒ (ii) If x_0 is a solution of pre-variational inequality (7), then

$$T(x_0)^t\, \tau(x, x_0) \geqslant f(x_0) - f(x), \qquad \forall x \in K.$$

Since $T(x_0) = -\nabla f(x_0)$, let $\eta(x) = \tau(x, x_0)$, $\forall x \in K$, and we have

$$f(x) - f(x_0) \geqslant \nabla f(x_0)^t\, \eta(x), \qquad \forall x \in K.$$

Therefore, $f(x)$ is a Type I function with respect to $\eta(x)$ at x_0.

(ii) ⇒ (i) If $f(x)$ is a Type I function with respect to $\eta(x)$ at x_0 and $\tau(x, x_0) = \eta(x)$, then

$$f(x) - f(x_0) \geqslant \nabla f(x_0)^t\, \eta(x), \qquad \forall x \in K.$$

Therefore

$$T(x_0)^t\, \tau(x, x_0) \geqslant f(x_0) - f(x), \qquad \forall x \in K.$$

Then x_0 is a solution of pre-variational inequality (7).

THEOREM 7. *Let $x_0 \in K$ and let $f(x)$ be the Type I function with respect to $\eta(x)$ at x_0. If*

$$\nabla f(x_0) = 0,$$

then x_0 is a minimum of (P_2).

Proof. Since $f(x)$ is a Type I function with respect to $\eta(x)$ at x_0, then for any $x \in K$,

$$f(x) - f(x_0) \geqslant \nabla f(x_0)' \eta(x) = 0.$$

Therefore $f(x) \geqslant f(x_0)$ for all $x \in K$. It follows that x_0 is an optimal solution of (P_2).

4. An Existence Theorem for Pre-variational Inequalities

In this section the existence of the solution of the pre-variational inequality (8) is considered. Let K be a subset of R^n, $T: K \to R^n$, $\tau: K \times K \to R^n$, by which for each $x \in K$, $g(y) = \tau(y, x)$: $K \to R^n$ is continuous on K and $\tau(x, x) = 0$. Then the pre-variational inequality problem is

$$\text{Find } x \in K, \qquad \text{such that} \qquad T(x)' \tau(y, x) \geqslant 0, \qquad \forall y \in K. \tag{8}$$

T: $K \to R^n$ is said to be continuous on K if for any $\{x_i\} \subset K$, $x_i \to x \in K$, then $\|T(x_i) - T(x)\| \to 0$ $(i \to \infty)$.

THEOREM 8. *Let K be a bounded closed convex set of R^n. Assume that T and τ satisfy the following conditions*:

(i) *T is continuous on K;*

(ii) *$h(y) = T(x)' \tau(y, x)$ is convex for each fixed $x \in K$.*

Then the pre-variational inequality (8) *is solvable.*

Proof. Let

$$F(y) = \{x \in K \mid T(x)' \tau(y, x) \geqslant 0\}, \qquad \forall y \in K.$$

Let $x_1, ..., x_n \in K$, $\lambda_i \geqslant 0$ $(i = 1, ..., n)$, $\sum_{(n)} \lambda_i = 1$. Suppose that $x = \sum_{(n)} \lambda_i x_i \notin \bigcup F(x_i)$. Then

$$T(x)' \tau(x_i, x) < 0, \qquad i = 1, ..., n,$$

i.e.,

$$\sum_{(n)} \lambda_i T(x)' \tau(x_i, x) < 0.$$

By the convexity of h for fixed x, we get

$$h\left(\sum_{(n)} \lambda_i x_i\right) \leqslant \sum_{(n)} \lambda_i h(x_i),$$

i.e.,

$$T(x)' \tau\left(\sum_{(n)} \lambda_i x_i, x\right) \leqslant \sum_{(n)} \lambda_i T(x)' \tau(x_i, x).$$

Hence by $\tau(x, x) = 0$,

$$0 = T(x)' \tau\left(\sum_{(n)} \lambda_i x_i, x\right) \leqslant \sum_{(n)} \lambda_i T(x)' \tau(x_i, x) < 0.$$

It is a contradiction. So $x = \sum_{(n)} \lambda_i x_i \in \bigcup F(x_i)$.

It is obvious that $F(y)$ is a closed set for any y in K. Since K is bounded, $F(y)$ is a compact. By the KKM theorem [9], we obtain

$$\bigcap F(y) \neq \varnothing.$$

Let $x \in \bigcap F(y)$. Then

$$T(x)' \tau(y, x) \geqslant 0, \qquad \forall y \in K.$$

So x is a solution of pre-variational inequality (8). ∎

However, the condition that $h(y) = T(x)' \tau(y, x)$ be convex may be replaced by the requirements that $T(x)$ be positive and $\tau(y, x)$ be convex in its first variable, because these conditions imply that $h(y) = T(x)' \tau(y, x)$ is convex.

COROLLARY 1 (Hartman and Stampacchia [13]). *Let K be a bounded closed convex in R^n and T be continuous on K. Then there exists $x \in K$ such that*

$$T(x)' (y - x) \geqslant 0, \qquad \forall y \in K.$$

T is said to be *pre-coercive with respect to* $\tau(y, x)$ if there exists $x_0 \in K$ such that

$$(T(x) - T(x_0))' \tau(x, x_0)/\|\tau(x, x_0)\| \to +\infty, \tag{11}$$

whenever $\|x\| \to +\infty$.

If $\tau(y, x) = y - x$, this condition coincides with the coercive condition in Hartman and Stampacchia [13]: T is called coercive if there exists $x_0 \in K$ such that

$$(T(x) - T(x_0))' (x - x_0)/\|x - x_0\| \to +\infty, \tag{12}$$

whenever $\|x\| \to +\infty$.

Let $\tau'_1(x, x) = [d/dy](\tau(y, x))|_{y=x}$.

$\tau(y, x)$ *is normal at* $x \in K$ if $\tau'_1(x, x) > 0$; $\tau(y, x)$ is normal on K if $\tau(y, x)$ is normal at each $x \in K$.

THEOREM 9. *Let K be a closed convex set of R^n. Suppose that T and τ satisfy the following conditions*:

(1) *T is continuous on K and pre-coercive with respect to $\tau(y, x)$*;

(2) *$\tau(y, x)$ is normal on K*;

(3) *$h(y) = T(x)' \tau(y, x)$ is convex for any fixed $x \in K$.*

Then the pre-variational inequality (8) *is solvable.*

Proof. Let B_r denote the closed ball of centre 0 and radius r in R^n. Then the conditions in Theorem 8 can be satisfied if we substitute $K \cap B_r$ for K. Hence there exists a solution x_r for the pre-variational inequality:

$$\text{Find } x \in K \cap B_r, \qquad \text{such that} \qquad T(x)' \tau(y, x) \geqslant 0, \qquad \forall y \in K \cap B_r.$$

Choose $r \geqslant \|x_0\|$ with x_0 as in the pre-coercive condition. Then

$$T(x_r)' \tau(x_0, x_r) \geqslant 0. \tag{13}$$

Moreover,

$$\begin{aligned} &T(x_r)' \tau(x_0, x_r) \\ &\quad = (T(x_r) - T(x_0))' \tau(x_0, x_r) + T(x_0)' \tau(x_0, x_r) \\ &\quad \leqslant -(T(x_0) - T(x_r))' \tau(x_0, x_r)) + \|T(x_0)\| \, \|\tau(x_0, x_r)\| \\ &\quad \leqslant \|\tau(x_0, x_r)\| (-(T(x_0) - T(x_r))' \tau(x_0, x_r)) / \|\tau(x_0, x_r)\| + \|T(x_0)\|). \end{aligned} \tag{14}$$

If $\|x_r\| = r$ for all r, we choose r large enough that (14) and condition (11) imply $T(x_r)' \tau(x_0, x_r) < 0$, which contradicts (13). Hence, there exists r such that $\|x_r\| < r$. For each $y \in K$, we choose $\varepsilon > 0$ small enough that $x_r + \varepsilon\tau(y, x_r) \in K \cap B_r$. Therefore

$$T(x_r)' \tau(x_r + \varepsilon\tau(y, x_r), x_r) \geqslant 0, \tag{15}$$

since

$$\begin{aligned} \tau(x_r + \varepsilon\tau(y, x_r), x_r) &= \tau(x_r, x_r) + \varepsilon\tau(y, x_r)\, \tau'_1(x_r, x_r) + o(\varepsilon) \\ &= \varepsilon\tau(y, x_r)\, \tau'_1(x_r, x_r) + o(\varepsilon). \end{aligned}$$

Thus

$$T(x_r)^t\,(\varepsilon\tau(y, x_r)\,\tau_1'(x_r, x_r) + o(\varepsilon)) \geqslant 0. \tag{16}$$

Dividing the inequality (15) by ε, then letting $\varepsilon \downarrow 0$, we have

$$T(x_r)^t\,\tau(y, x_r)\,\tau_1'(x_r, x_r) \geqslant 0.$$

Since $\tau_1'(x_r, x_r) > 0$, we have

$$T(x_r)^t\,\tau(y, x_r) \geqslant 0, \qquad \forall y \in K.$$

Therefore x_r is a solution of pre-variational inequality (8). ∎

COROLLARY 2 (Hartman and Stampacchia [13]). *Let K be a closed convex subset in R^n and T be continuous and coercive on K; then there exists $x \in K$ such that*

$$T(x)^t\,(y - x) \geqslant 0, \qquad \forall y \in K.$$

We call $\tau(y, x)$ *regular at* x if for each $y \in K$, there exists a positive constant $\alpha = \alpha(x, y)$,

$$\lim_{\varepsilon \downarrow 0} \tau(x + \varepsilon y, x)/\varepsilon \to \alpha y. \tag{17}$$

$\tau(y, x)$ *is regular on* K if $\tau(y, x)$ is regular at each $x \in K$.

THEOREM 10. *Let K be a closed convex set of R^n. Suppose T and τ satisfy the following conditions:*

(1) *T is continuous and pre-coercive with respect to $\tau(y, x)$;*

(2) *$\tau(y, x)$ is regular on K;*

(3) *$h(y) = T(x)^t\,\tau(y, x)$ is convex for any fixed $x \in K$.*

Then the pre-variational inequality (8) *is solvable.*

Proof. From the proof of Theorem 9, there exists a solution x_r of the pre-variational inequality

Find $x \in K \cap B_r$, such that $T(x)^t\,\tau(y, x) \geqslant 0, \quad \forall y \in K \cap B_r.$

For each $y \in K$, we choose $\varepsilon > 0$ sufficiently small, r sufficiently large ($\|x_r\| < r$) that

$$T(x_r)^t\,\tau(x_r + \varepsilon\tau(y, x_r), x_r) \geqslant 0. \tag{18}$$

Since $\tau(y, x)$ is regular on K (see (17)), divide the inequality (18) by ε, then let $\varepsilon \downarrow 0$, and there exists $\alpha(y, x_r) > 0$ such that

$$\alpha(y, x_r)\, T(x_r)'\, \tau(y, x_r) \geqslant 0.$$

Thus

$$T(x_r)'\, \tau(y, x_r) \geqslant 0, \qquad \forall y \in K.$$

The proof is completed. ∎

5. Conclusion

In this paper, we discussed a class of nonconvex nonsmooth functions (semi-preinvex functions) which includes preinvex functions and arc-connected convex functions if $\tau(y, x, \alpha)$ is suitably chosen. We derived the Fritz-John condition by using an alternative theorem for the semi-preinvex program and introduced the pre-variational inequality, which is a necessary condition for the optimal solution. We also proved some existence theorems by using the Knaster–Kuratowski–Mazurkiewcz theorem.

Recently, J. Parida, M. Sahoo, and A. Kumar [15] also discussed the pre-variational inequality (calling the pre-variational inequality the variational-like inequality) and proved the existence of the solution of the variational-like inequality by using the Kakutani fixed point theorem. However, in our approach, we used a different method to prove the existence of the pre-variational inequality, in which the useful coercive condition is generalized.

Acknowledgments

The authors are indebted to one of the referees for his considerable editorial assistance, which resulted in the final presentation of the paper.

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